On finite groups with formational basic subgroups of fans of Sylow subgroups

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Only finite groups are considered. Let D be a subgroup of a group G. A subgroup H is called *intermediate* for D if $D \le H \le G$.

Definition [1]. A system $\{G_\alpha, N_\alpha\}$ of intermediate for D subgroups G_α and their normalizers $N_\alpha = N_G(G_\alpha)$ is called a *fan* of the subgroup D if for each intermediate subgroup H there exists a unique index α such that $G_\alpha \le H \le N_\alpha$. The subgroups G_α are called the *basic* subgroups of the fan. If there exists a fan of D, then D is called a *fan subgroup* of G.

In [**1**] it is established that any Sylow subgroup of a group is a fan subgroup.

Recall [**2**, Example IV.3.4(g)] that $\mathfrak{T}_\prec$ is the class of all Sylow tower groups of type $\prec$, where $\prec$ is an arbitrary linear ordering on the set $\mathbb{P}$ of all primes, i.e.

$\mathfrak{T}_\prec = (G \mid |G| = p_1^{m_1} p_2^{m_2} \cdots p_r^{m_r}$, where p_i is prime, $p_1 \prec p_2 \prec \cdots \prec p_r$, and G has a normal subgroup of the order $p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$ for $k = 1, 2, \dots, r)$.

Denote by $\mathfrak{N}^2$ the class of all metanilpotent groups.

Theorem. *Let $\mathfrak{F}$ be a hereditary saturated formation such that $\mathfrak{F} \subseteq \mathfrak{T}_\prec \cap \mathfrak{N}^2$. A group $G \in \mathfrak{F}$ if and only if every basic subgroup of the fan of every Sylow subgroup of G belongs to $\mathfrak{F}$.*

Corollary 1. *A group G is a Sylow tower group of type $\prec$ and metanilpotent if and only if every basic subgroup of the fan of every Sylow subgroup of G is a Sylow tower group of type $\prec$ and metanilpotent.*

Corollary 2. *A group G is a Sylow tower group of type $\prec$ and has a nilpotent derived subgroup if and only if every basic subgroup of the fan of every Sylow subgroup of G is a Sylow tower group of type $\prec$ and has a nilpotent derived subgroup.*

Corollary 3 [3]. *A group G is supersoluble if and only if every basic subgroup of the fan of every Sylow subgroup of G is supersoluble.*

Note that the group G is not always a basic subgroup of the fan of some its Sylow subgroup. For example, let $G = HK$, where the subgroup H is Z_7 and the subgroup K is equal to $\mathrm{Aut}(Z_7) \cong Z_6$. It is easy to see that the set of all basic subgroups of the fans of all Sylow subgroups of G consists of $\mathrm{Syl}(G)$ and subgroups of orders 14 and 21.

References

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