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A PROBLEM OF WAVE PROPAGATION IN AN ELASTIC TUBE CONTAINING HETEROGENEOUS LIQUID

In the work there is performed the solution of a one-dimensional problem of harmonic waves propagation in an orthotropic elastic tube containing heterogeneous incompressible liquid with the rheological behaviour described by Maxwell model. There is numerically depicted the influence of concentration of inclusions on wave characteristics for the case of long stationary waves propagation in heterogeneous liquid flowing in an elastic tube of variable circular section. The properties of the liquid comply with linear visco-elastic model of Foigt. The solution of this problem is defined by singular boundary problem of Sturm – Liouville. It is assumed that the tube is rigidly fixed to surrounding and, thus, its longitudinal displacement is equal to null. Cases of finite and semi-infinite tubes are considered.

Keywords: wave propagation, elastic tube, heterogeneous liquid, harmonic waves, viscoelasticity, haemodynamics.

1 Introduction. The flow of liquid in deformed tubes in many cases can be defined by equations of hydraulic approximation. The majority of works in this direction are based on the assumption of homogeneity of tube material. In many practically important cases we have to deal with propagation of stationary waves in elastic tubes that contain heterogeneous liquids and the velocity of propagation is a wave local parameter and it is considered as a function of coordinates.

For the analysis of flow of colloidal solutions, suspensions, high-molecular compounds we use rheological models representing different combinations of elastic and viscous elements. Their behavior at least qualitatively corresponds to the behavior of the abovementioned mediums.

Theoretical developments, obtained from solutions of interaction problems for a cylindrical shell with a viscous liquid flowing in it due to the definite physical approximations may be carried for the case of disperse liquid. This generalization is made through introduction of the dynamic viscosity effective coefficient.

From the qualitative analysis it follows that at known boundary conditions (functional systems) liquid rheological properties have an apparent impact on its velocity and hydraulic impedance, and viscosity of tube material affects the displacement, velocity and impedance.

It should be noted that due to the problem of linearity the real parts of solutions, obtained from the arbitrary kernels of heredity, have their physical meaning.

2 Mathematical formulation. Firstly, there should be defined the system that describes propagation of small amplitude waves in a suspension flowing in a deformed shell. A liquid mathematical model should consider the fact that multiphase systems are mixtures of hard particles, liquefied droplets and bubbles (discrete phases) that are widespread in a liquid (carrying and continuous phase) [1].

Research of multiphase systems dynamics covers wide fields of science and technology and it is connected with a lot of fundamental problems. Here, we can mention, e. g. such important cases as cryogenic liquids pumping-over, radioactive precipitation, deposition, haemodynamics etc. For our purposes we will interpret disperse medium as incompressible Newtonian liquid with the density of water ρ_f, containing non-interacting particles of identical size. It is assumed that the velocities of continuous and discrete phases are the same. Then, an effective dynamic viscosity μ of diluted suspension of hard spherical particles having neutral buoyancy (i. e. non-depositing and non-emerging) in a liquid carry-over with a viscosity of μ_0 can be calculated through the formula of Einstein [2]

$$\mu = \mu_0\left(1 + \frac{5}{2}\phi\right), \tag{1}$$

where ϕ – volumetric concentration of particles in parts of units. This result was generalized by Taylor [2] on suspension of droplets which keep their spherical form, e. g. due to surface tension. A consecutive correlation is as follows:

$$\mu = \mu_0\left\{1 + \phi\left(\frac{\mu_0 + 5\bar{\mu}/2}{\mu_0 + \bar{\mu}}\right)\right\}, \tag{2}$$

in which $\bar{\mu}$ – viscosity of liquid that makes droplets. When $\bar{\mu}$ becomes infinitely large, i. e. when the droplets appear to become, actually, hard particles, this correlation is reduced to (1).

Effective viscosity of hard asymmetric particles suspension increases with the growth of particles concentration as well as with power of their asymmetry. This dependence is defined by the formula

$$\mu = \mu_0\left(1 + K\phi\right),$$

where K (factor of geometry) is more than 5/2. In the case of rotation in relation to half-axles 6:1 of hard mixtures of non-spherical particles of elliptical form, K takes the value equal to 5 and the mixture viscosity increases as follows [2]:

$$\mu = \mu_0\left(1 + 5\phi\right). \tag{3}$$

The assumptions provide opportunity to consider the known contact conditions of conjugation of linear hydroelasticity. Now if to take into consideration the condition of impermeability and to assume that the tube is rigidly fixed to surroundings and the wall material cannot make any movement along its axis x, then there can be written the mean equations of impermeability and those of Navier-Stokes for the mixture as a whole based on abovementioned assumptions in the following form [3]:

$$\frac{\partial u}{\partial x} + \frac{2}{R}\frac{\partial w}{\partial t} = 0; \tag{4}$$

$$\frac{1}{\rho_f}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial t} + \frac{8\mu}{\rho_f R^2}u = 0. \tag{5}$$

In (4), (5) $w(x, t)$ – is a radial displacement of a tube of radius R and thickness h, $u(x, t)$ – is a mean velocity of mixture flow, $p(x, t)$ – hydrodynamic pressure. As for the dynamic coefficient of mixture viscosity μ, it depends on concentration ϕ and should be defined in actual examples by formulas (1)–(3).

The systems (4) and (5) can be transformed to a single equation

$$\frac{1}{\rho_f}\frac{\partial^2 p}{\partial x^2}+\frac{8\mu}{\rho_f R^2}\frac{\partial u}{\partial x}-\frac{2}{R}\frac{\partial^2 w}{\partial t^2}=0.$$

Substituting here $\frac{\partial u}{\partial x}$ for $\frac{2}{R}\frac{\partial w}{\partial t}$ and putting this into the last dependence, we have

$$\frac{1}{\rho_f}\frac{\partial^2 p}{\partial x^2}-\frac{16\mu}{\rho_f R^3}\frac{\partial w}{\partial t}-\frac{2}{R}\frac{\partial^2 w}{\partial t^2}=0\,. \tag{6}$$

Next, for completion of equation (6), let's write down the equation of condition for the tube material, considering that it is elastic, orthotropic and thin-walled. For these conditions it is sufficient to use the following correlation [4]:

$$p=\frac{E_2}{1-v_1 v_2}\frac{h}{R^2}w+\rho_* h\frac{\partial^2 w}{\partial t^2}\,, \tag{7}$$

where ρ_* – is a density of tube material, v_1 and v_2 – are coefficients of Poisson, E_2 – modulus of elasticity in a circumferential direction. It should be mentioned that the condition of Maxwell to hold herewith:

$$E_2 v_1 = E_1 v_2\,,$$

E_1 – is an axial Young's modulus.

Let's take the second derivative along x in the equation (7) and consider the result in (6). Then, considering

$$c_0^2=\frac{E_2}{2\rho_f\left(1-v_1 v_2\right)}\frac{h}{R} \quad \text{and} \quad \frac{\rho_*}{\rho_f}=\rho,$$

we get the following equation

$$\frac{\partial^2 w}{\partial x^2}+\rho\frac{Rh}{2c_0^2}\frac{\partial^4 w}{\partial x^2\partial t^2}-\frac{8\mu}{\rho_f c_0^2 R^2}\frac{\partial w}{\partial t}-\frac{1}{c_0^2}\frac{\partial^2 w}{\partial t^2}=0\,, \tag{8}$$

which describes dynamic behaviour of the "shell-liquid" system.

3 Numerical method and parameters. The method of Fourier is applied for description of complex impulses, typical for wave processes; hence solution of equation (8) is obtained from the final sum of the main oscillation and higher harmonics. [5] This statement allows to represent the function w in the following form:

$$w=\sum_{s=1}^{S} y_s\left(x\right)\exp(is\omega t)\,, \tag{9}$$

where $y_s(x)$ – unknown complex functions of coordinates of dimension, ω – known angular frequency, i – imaginary unit, and S – harmonic value.

Considering the system linearity, the change of each harmonics s and the definition of the disturbance form there can be obtained the sum of each component related to the given point. Substituting (9) into (8), for s^{th} harmonics we have

$$y_s'' + \lambda_s^2 y_s = 0 \,, \tag{10}$$

where the prime means differentiation along coordinate x, and value λ_s is defined from the solution of the following disperse equation:

$$\lambda_s^2 = \frac{\dfrac{\omega}{c_0^2} s \left\{ \omega s - i \dfrac{8\mu}{\rho_f R^2} \right\}}{1 - \rho \dfrac{Rh}{2c_0^2} \omega^2 c_0^2} \,. \tag{11}$$

Dividing equation (11) into real and imaginary parts and introducing the indications

$$\lambda_{0s} = \frac{\dfrac{s^2\omega^2}{c_0^2}}{1 - \rho \dfrac{Rh}{2c_0^2} \omega^2 s^2} \,, \qquad \lambda_{1s} = \frac{\dfrac{s^2\omega^2}{c_0^2} \dfrac{8\mu}{\rho_f R^2}}{1 - \rho \dfrac{Rh}{2c_0^2} \omega^2 s^2}$$

we get

$$\lambda_s^2 = \lambda_{0s} - i\lambda_{1s} \,. \tag{12}$$

Solving disperse equation (12), choosing the root $\operatorname{Im} \lambda < 0$ and using the known formula of calculation of square root from a complex value, we find:

$$\lambda_s = \delta_{0s} - i\delta_{1s} \tag{13}$$

In (13) it is assumed that

$$\delta_{0s} = \left\{ \frac{1}{2} (\lambda_{0s} + m) \right\}^{\frac{1}{2}}, \ \delta_{1s} = \left\{ \frac{1}{2} (m_s - \lambda_{0s}) \right\}^{\frac{1}{2}}, \text{ and } m_s = \left(\lambda_{0s}^2 + \lambda_{1s}^2 \right)^{\frac{1}{2}}.$$

With this the velocity of propagation of s^{th} wave is defined as $\dfrac{s\omega}{\delta_{0s}}$, and δ_{1s} – damping coefficient.

4 Derivations and numerical analysis. Firstly, it should be mentioned that the common solution to the equation (10) is written in the form:

$$y_s = A_s e^{-i\lambda_s x} + B_s e^{i\lambda_s x} \,, \tag{14}$$

where A_s and B_s – constants of integration determined from the boundary conditions to be defined further. Now for function w and p we may write

$$w = \sum_{s=1}^{S} \left\{ A_s e^{-i\lambda_s x} + B_s e^{i\lambda_s x} \right\} \exp(is\omega t) \tag{15}$$

and

$$p = \left\{ \frac{E_2}{1 - v_1 v_2} \frac{h}{R^2} - \rho_* h \omega^2 \right\} \sum_{s=1}^{S} \left\{ A_s e^{-i\lambda_s x} + B_s e^{i\lambda_s x} \right\} \exp(is\omega t) \ . \tag{16}$$

Both of these results can be derived from formulas (7) and (9), in case we consider in them the dependence (14).

Now it is remained to define the velocity of liquid flow. For this we state

$$u = \sum_{s=1}^{S} U_s \left(x \right) \exp(is\omega t) \ . \tag{17}$$

With this in mind and using the equation (5), after elementary conversions it is possible to find

$$u = -i \left\{ \frac{E_2}{1 - v_1 v_2} \frac{h}{R^2} - \rho_* h \omega^2 \right\} \sum_{s=1}^{S} \frac{\lambda s}{I_s} \left(-A_s e^{-i\lambda_s x} + B_s e^{i\lambda_s x} \right) \exp(is\omega t) \ , \tag{18}$$

where the value

$$I_s = \frac{-\left(\dfrac{\partial p_s}{\partial x} \right)}{u_s} = \rho_f \left(is\omega + \frac{8\mu}{\rho_f R^2} \right)$$

is a hydraulic impedance of s^{th} harmonic. The value of $8\mu/R^2$ characterizes hydraulic resistance, and $s\omega$ – induction. Hence it follows that hydraulic resistance linearly depends on μ, and induction – on harmonic s.

Now, let's describe pressure propagation, flow velocity and displacement for a straight tube of length l. For this let's formulate the following boundary conditions. The pressure changes according to the law

$$\sum_{s=1}^{S} p_{0s} \exp(is\omega t) \tag{19}$$

at $x = 0$ and for simplicity it equals to zero at $x = l$. In (19) p_{0s} – known empirical constants. Due to the written boundary conditions, the obtained system of algebraic equations is necessary for definition of A_s and B_s. It has the following form:

$$\alpha \left(A_s + B_s \right) = p_{0s} ,$$

$$A_s e^{-i\lambda_s l} + B_s e^{i\lambda_s l} = 0 ,$$

$$\alpha = \frac{E_2}{1 - v_1 v_2} \frac{h}{R^2} - \rho_* h \omega^2 \ .$$

Hence it follows that

$$A_s = -i\frac{p_{0s}e^{i\lambda_s l}}{2\alpha \sin\lambda_s l} \text{ and } B_s = i\frac{p_{0s}e^{-i\lambda_s l}}{2\alpha \sin\lambda_s l}.$$

Using these equations in (15), (16) and (18), we find

$$w(x,t) = -\frac{1}{\alpha}\sum_{s=1}^{S} p_{0s}\frac{\sin\lambda_s(x-l)}{\sin\lambda_s l}\exp(is\omega t); \tag{20}$$

$$p(x,t) = -\sum_{s=1}^{S} p_{0s}\frac{\sin\lambda_s(x-l)}{\sin\lambda_s l}\exp(is\omega t); \tag{21}$$

$$u(x,t) = \sum_{s=1}^{S}\frac{\lambda_s}{I_s}p_{0s}\frac{\cos\lambda_s(x-l)}{\sin\lambda_s l}\exp(is\omega t). \tag{22}$$

Consecutive correlations for the limited case of semi-infinite tube can be obtained through calculation of limit of the expressions (20)–(22) at l approaching infinity. It may be shown that at Im $\lambda_s < 0$ (that was mentioned earlier) and

$$\lim_{l\to\infty}\frac{\sin\lambda_s(x-l)}{\sin\lambda_s l} = -e^{-i\lambda_s x};$$

$$\lim_{l\to\infty}\frac{\cos\lambda_s(x-l)}{\sin\lambda_s l} = -ie^{-i\lambda_s x},$$

then from the above mentioned formulas it follows that the related solution can be written as follows:

$$w(x,t) = \frac{1}{\alpha}\sum_{s=1}^{S} p_{0s}e^{-i\lambda_s x}\exp(is\omega t); \tag{23}$$

$$p(x,t) = \sum_{s=1}^{S} p_{0s}e^{-i\lambda_s x}\exp(is\omega t); \tag{24}$$

$$u(x,t) = i\sum_{s=1}^{S}\frac{\lambda_s}{I_s}p_{0s}e^{-i\lambda_s x}\exp(is\omega t). \tag{25}$$

It should be noted that due to the system linearity, the physical meaning has the true parts of the built solution.

Let's move forward to calculation of the amplitude of pressure $|p_s|$ for the s^{th} harmonic. We have

$$p_s = p_{0s}e^{-i\lambda_s x}e^{is\omega t},$$

hence, taking into account (13) and considering Euler's formula we may write

$$p_s = p_{0s}e^{-i\delta_{0s}x}e^{-\delta_{1sx}}\{\cos(s\omega t) + i\sin(s\omega t)\}.$$

From the previous equation it is easy to get for pressure amplitude that

$$|p_s| = p_{0s} e^{-\delta_{1s} x} . \tag{26}$$

Following the equation (23) for the amplitude of displacement we may write

$$|w_s| = \frac{p_{0s}}{\alpha} e^{-\delta_{1s} x} . \tag{27}$$

Doing the same we can calculate the amplitude of flow velocity, that for the s^{th} harmonic has the following appearance:

$$|u_s| = p_{0s} e^{-\delta_{1s} x} \frac{\sqrt{a_{1s} + a_{2s}}}{a_{0s}} . \tag{28}$$

Here the coefficients a_{0s}, a_{1s} and a_{2s} to be written down as follows:

$$\alpha_{0s} = \frac{64\mu^2}{R^4} + \rho_f^2 s^2 \omega^2 , \ \alpha_{1s} = \frac{8\mu}{R^2} \delta_{1s} + \rho_f s \omega \delta_{0s} , \ \alpha_{2s} = \frac{8\mu}{R^2} \delta_{0s} - \rho_f s \omega \delta_{1s} .$$

5 Results and Conclusions. For estimation of the result, received form considering an "amendment" to the dynamic viscosity coefficient, we are interested to see the influence of heterogeneity. In order to get a numerical result, we assume that the tube is orthotropic. The numerical experiment with the following system parameters is proposed $R = 0.5$ cm; $\bar{\mu} = 0.1$ g/cm · sec; $h = 0.2$ cm; $\rho_f = \rho_* = 1$ g/cm^3; $\omega = 2\pi$ sec^{-1}; $x = 0$; $E_2 = 4 \cdot 10^6$ dyn/cm^2; $\nu_1 = 0.1$; $\nu_2 = 0.3$; $p_{01} = 1.4 \cdot 10^3$ dyn/cm^2; $p_{03} = 2.4 \cdot 10^2$ dyn/cm^2.

Table 1 shows the values of wave velocity depending on concentration ϕ, at $s = 1$ and $s = 3$, when an effective viscosity is calculated through the formula (1). Table 2 shows the values for the damping coefficient depending on ϕ. Table 3 for the same case shows the dependence of velocity amplitude of mixture flow for various volumetric concentrations.

Table 1 – **The values of wave velocity depending on concentration ϕ**

s	c, cm/sec			
	$\phi = 0$	$\phi = 0.1$	$\phi = 0.2$	$\phi = 0.3$
1	882	855	824	793
3	895	889	905	901

Table 2 – **The values for the damping coefficient depending on ϕ**

s	δ_1, sec^{-1}			
	$\phi = 0$	$\phi = 0.1$	$\phi = 0.2$	$\phi = 0.3$
1	0.0017	0.0025	0.0038	0.0039
3	0.0018	0.0026	0.0035	0.0043

Table 3 – **Velocity amplitude of mixture flow for various volumetric concentrations**

s	$\lvert u_s \rvert$, cm/sec			
	$\phi = 0$	$\phi = 0.1$	$\phi = 0.2$	$\phi = 0.3$
1	1.43	1.34	1.25	1.17
3	0.26	0.25	0.25	0.24

Based on received numerical calculations we can make the following conclusions:

– the wave velocity and the amplitude of flow velocity decrease with the increase of ϕ;

– the biggest increase against concentration ϕ is observed for the coefficient δ_1 (almost two times more);

– with the increase of harmonics the wave velocity increases.

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О РАСПРОСТРАНЕНИИ ВОЛН В УПРУГОЙ ТРУБКЕ, СОДЕРЖАЩЕЙ ГЕТЕРОГЕННУЮ ЖИДКОСТЬ

Выполнено решение одномерной задачи о распространении гармонических волн в ортотропной упругой трубке, содержащей неоднородную несжимаемую жидкость, реологическое поведение которой описывается моделью Максвелла. Численно проанализировано влияние концентрации включений на волновые характеристики для случая распространения длинных стационарных волн в неоднородной жидкости, текущей в упругой трубке переменного округлого сечения, свойства которой соответствуют линейной вязкоупругой модели Фойхта. Решение этой задачи определяется сингулярной краевой задачей Штурма – Лиувилля. Предполагается, что трубка жестко закреплена в окружающем пространстве и, следовательно, ее продольное смещение равно нулю. Рассмотрены случаи конечных и полубесконечных трубок.

Ключевые слова: распространение волн, упругая трубка, неоднородная жидкость, гармонические волны, вязкоупругость, гемодинамика.

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