## On classes of finite groups in which maximal subgroups are $k$-submodular

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All groups under consideration are finite. In a nilpotent group, every subgroup is subnormal. From [**1**, Theorem VI.9.2(b)] it follows that if the group $G$ is supersoluble, then every its proper subgroup can be connected to $G$ by a chain of subgroups of prime indices. Schmidt proved in [**2**] that the subgroup $M$ of the group $G$ is a maximal modular subgroup in $G$ if and only if either $M$ is maximal normal subgroup of $G$, or $G/\mathrm{Core}_G(M)$ is a non-abelian group of order $pq$ for some primes $p$ and $q$.

**Definition 1.** *Let $n$ be a natural number. A subgroup $H$ of a group $G$ will be called $n$-modularly embedded in $G$ if either $H \trianglelefteq G$ or $H \neq \mathrm{Core}_G(H)$, $|G : H| = p$ and $|G/\mathrm{Core}_G(H)| = pq^n$, $q^n$ divides $p-1$ for some primes $p$ and $q$.*

**Definition 2.** *Let $k$ be a fixed natural number. A subgroup $H$ of a group $G$ will be called $k$-submodular in $G$ if there exists a chain of subgroups $H = H_0 \leq H_1 \leq \cdots \leq H_{m-1} \leq H_m = G$ such that $H_{i-1}$ is $n$-modularly embedded in $H_i$ for some natural $n \leq k$ and every $i = 1, \ldots, m$.*

Note if $n = 1$ and a maximal subgroup in $G$ is $n$-modularly embedded, then it is modular. If $k = n = 1$ and a subgroup in $G$ is $k$-submodular, then it is submodular in $G$. Groups with given systems of submodular subgroups were studied in [**3**]–[**6**].

**Theorem.** *Let $\mathfrak{X}$ be a class of all groups $G$ in which any maximal subgroup is $k$-submodular. Let $\mathfrak{Y}$ be a class of all groups $G$ such that every maximal subgroup of any subgroup $A$ of $G$ is $k$-submodular in $A$. Then*

*(1) $\mathfrak{X}$ is a Schunk class consisting of supersoluble groups.*

*(2) $\mathfrak{Y}$ is a hereditary formation.*

*(3) The group $G \in \mathfrak{X}$ if and only if every $G/\Phi(G) \in \mathfrak{H}$.*

The properties of $k$-submodular subgroups of groups are obtained [**7**].

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### References

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